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COMMENT

Canonical and Feynman quantization in Riemannian spaces†

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Abstract. A paper by Ben-Abraham and Lonke claims to resolve the ambiguity between the Shrödinger equation found by canonical quantization and that found from the Feynman propagator. It is shown that this claim is ill founded.

1. Introduction

A paper by Cheng (1972) derives a result, originally claimed by Dewitt (1957), to the effect that the Shrödinger equation obtained from the Feynman propagator differs from that obtained through direct canonical quantization by a term $R/6$, where R is the scalar curvature. The approach of both Cheng and Dewitt is to normalize the wavefunction by

$$\int d^n q g^{1/2} |\psi|^2 = 1.$$

A subsequent publication by Ben-Abraham and Lonke (1973, to be referred to as BAL) claims that, if the wavefunction is normalized without the factor $g^{1/2}$ and care is taken to ensure that all integrals are scalars, the two forms of the Shrödinger equation agree.

At the risk of overburdening the literature, it is worthwhile to point out errors in BAL and to present the calculation correctly for the following two reasons: the normal coordinates used in BAL bring clarity and simplicity to the calculation; the normalization used by BAL is the more symmetrical.

2. The canonical Shrödinger equation in normal coordinates

The normalization used in BAL requires that the wavefunction is a tensor density of weight 1/2 and that the Shrödinger equation obtained from canonical quantization:

$$i\frac{\partial \psi}{\partial t} = -\tfrac{1}{2} g^{-1/4} \frac{\partial}{\partial q^\alpha} g^{1/2} g^{\alpha\beta} \frac{\partial}{\partial q^\beta} g^{-1/4} \psi \tag{2.1}$$

contains the correct covariant Laplacian. The change to normal coordinates $^*q^\alpha$ with

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origin at $\underset{0}{q}^\alpha$ is effected by the implicit transformation (Veblen 1952)

$$q^\alpha - \underset{0}{q}^\alpha = {}^*q^\alpha - \frac{1}{2!}\underset{0}{\Gamma}{}^\alpha_{\beta\gamma}{}^*q^\beta{}^*q^\alpha - \frac{1}{3!}\left(\frac{\partial\Gamma^\alpha_{\beta\gamma}}{\partial q^\lambda}\right)_0{}^*q^\beta{}^*q^\gamma{}^*q^\lambda + \ldots$$

where $\Gamma^\alpha_{\beta\gamma}$ is the Christoffel symbol. Normal coordinates are particularly useful because at the origin the transformed connections satisfy

$${}^*\underset{0}{\Gamma}{}^\alpha_{\beta\gamma} = 0; \qquad \left(\frac{\partial^*\Gamma^\alpha_{\beta\gamma}}{\partial^*q^\lambda} + \frac{\partial^*\Gamma^\alpha_{\gamma\lambda}}{\partial^*q^\beta} + \frac{\partial^*\Gamma^\alpha_{\lambda\beta}}{\partial^*q^\gamma}\right)_0 = 0.$$

At the origin of the normal coordinate system the Laplacian becomes

$$\left(\frac{1}{2}{}^*g^{\alpha\beta}\frac{\partial^2}{\partial^*q^\alpha\partial^*q^\beta} - \frac{1}{4}{}^*g^{\alpha\beta}\frac{\partial^2 \ln {}^*g}{\partial^*q^\alpha\partial^*q^\beta}\right)_0.$$

This may be compared with the curvature scalar R at the origin

$$\underset{0}{R} = \left(-\frac{3}{2}{}^*g^{\alpha\beta}\frac{\partial^2 \ln {}^*g}{\partial^*q^\alpha\partial^*q^\beta}\right)_0.$$

The Shrödinger equation thus reduces to

$$\mathrm{i}\frac{\partial^*\underset{0}{\psi}}{\partial t} = -\frac{1}{2}{}^*\underset{0}{g}{}^{\alpha\beta}\left(\frac{\partial^2\psi}{\partial^*q^\alpha\partial^*q^\beta}\right)_0 - \frac{1}{12}\underset{0}{R}{}^*\underset{0}{\psi}. \tag{2.2}$$

(Note that ${}^*\psi$ does not denote complex conjugate. Since the wavefunction is a tensor density it transforms with the coordinates.)

Formally integrating (2.1) so as to exponentiate the scalar Laplacian

$$\psi(-\tau) = g^{1/4}\exp\left(-\mathrm{i}\frac{\tau}{2}g^{-1/2}\frac{\partial}{\partial q^\alpha}g^{1/2}g^{\alpha\beta}\frac{\partial}{\partial q^\beta}\right)g^{-1/4}\psi(0)$$

and inserting a delta function, the propagator can be seen to be

$$K(\underset{0}{q}, q; -\tau) = \underset{0}{g}^{1/4}\exp\left(-\mathrm{i}\frac{\tau}{2}\underset{0}{g}^{-1/2}\frac{\partial}{\partial \underset{0}{q}^\alpha}\underset{0}{g}^{1/2}\underset{0}{g}^{\alpha\beta}\frac{\partial}{\partial \underset{0}{q}^\beta}\right)\underset{0}{g}^{-1/2}\delta(\underset{0}{q} - q)g^{1/4},$$

a bi-half tensor density.

3. From propagator to Shrödinger equation

The small-time Feynman propagator from q at time $-\tau$ to $\underset{0}{q}$ at time zero is

$$K(\underset{0}{q}, q; -\tau) = (2\pi\mathrm{i}\tau)^{-n/2}\underset{0}{g}^{1/4}\exp(\mathrm{i}S)g^{1/4}$$

where S is the classical action

$$S = \min_{\{q(t)\}}\int_{-\tau}^{0} \mathrm{d}t\, L(q(t)); \qquad L = \tfrac{1}{2}g_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta.$$

The propagator clearly has the correct tensor symmetry.

The minimum of S is given by a geodesic path, along which L is a constant. In normal coordinates the geodesic is by design simple:

$$ {}^*q^\alpha(t) = -{}^*q^\alpha t/\tau $$

and so

$$ S = \frac{1}{2\tau} {}^*\underset{0}{g}_{\alpha\beta} {}^*q^\alpha {}^*q^\beta. $$

(This is the first mistake in BAL.) The infinitesimal evolution of the wavefunction is given by

$$ {}^*\underset{0}{\psi}(0) = (2\pi \mathrm{i}\tau)^{-n/2} {}^*\underset{0}{g}^{1/4} \int \mathrm{d}^n {}^*q \exp\left(\frac{\mathrm{i}}{2\tau} {}^*\underset{0}{g}_{\alpha\beta} {}^*q^\alpha {}^*q^\beta\right) {}^*g^{1/4} {}^*\psi(-\tau). \tag{3.1} $$

The function ${}^*g^{1/4} {}^*\psi(-\tau)$ can be expanded in a Taylor series:

$$ {}^*g^{1/4} {}^*\psi(-\tau) = {}^*\underset{0}{g}^{1/4} {}^*\underset{0}{\psi}(-\tau) + \left(\frac{\partial^* \psi(-\tau)}{\partial^* q^\alpha}\right)_0 $$
$$ + \frac{1}{2} {}^*\underset{0}{g}^{1/4} \left(\frac{\partial^{2*}\psi(-\tau)}{\partial^* q^\alpha \partial^* q^\beta} + \frac{1}{4} \frac{\partial^2 \ln {}^*g}{\partial^* q^\alpha \partial^* q^\beta}\right)_0 {}^*q^\alpha {}^*q^\beta + \ldots. $$

(This is the second departure from BAL where it is claimed that ${}^*g^{1/4}$ is a constant. Only the first derivative of g is zero. The Taylor expansion of any function of ${}^*q$ may always be made; the derivatives are not covariant as claimed in BAL. Dowker (1974) gives the correct covariant expansion of a scalar. Apart from the fact that Dowker uses the same normalization as Cheng the method employed here is the 'more efficient method' referred to by Dowker.)

The range of integration of ${}^*q^\alpha$ is $-\infty$ to $+\infty$ even if the variable runs over a finite interval of the real line, as for example S^1, the unit circle. This is to allow for classical paths which loop the circle infinitely many times. Using the identity

$$ \int_{-\infty}^{\infty} \mathrm{d}^n {}^*q \exp\left(\frac{\mathrm{i}}{2\tau} {}^*\underset{0}{g}_{\alpha\beta} {}^*q^\alpha {}^*q^\beta\right) {}^*q^{\alpha_1} {}^*q^{\alpha_2} \ldots {}^*q^{\alpha_{2m}} $$
$$ = (2\pi \mathrm{i}\tau)^{n/2} (\mathrm{i}\tau)^m {}^*\underset{0}{g}^{1/2} \sum_\sigma {}^*g^{\alpha_{\sigma_1}\alpha_{\sigma_2}} {}^*g^{\alpha_{\sigma_3}\alpha_{\sigma_4}} \ldots {}^*g^{\alpha_{\sigma_{2m-1}}\alpha_{\sigma_{2m}}}, $$

where the sum is over all permutations σ (integrals with odd numbers of ${}^*q$ are zero), the Shrödinger equation can be deduced from (3.1) by expanding the left-hand side in a Taylor series in τ and equating the first coefficient of τ:

$$ \mathrm{i}\frac{\partial \underset{0}{\psi}}{\partial t} = -\frac{1}{2} {}^*\underset{0}{g}^{\alpha\beta} \left(\frac{\partial^{2*}\psi}{\partial^* q^\alpha \partial^* q^\beta}\right)_0 + \frac{1}{12} \underset{0}{R}^* \underset{0}{\psi}. \tag{3.2} $$

4. Conclusion

The difference between the two equations (3.2) and (2.2) is a term $(R/6)^*\psi$. Thus, contrary to the claim of BAL and as might have been expected, changing the normalization of the wavefunction does not alter anything.

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